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# Dependence systems with the operator–image exchange property

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**Abstract**

An operator on a set $S$, i.e. an extensive and monotone (but not necessarily idempotent) function on the power set of $S$, generalizes the familiar notion of closure operator (transitive operator). This is one of several equivalent ways to define a dependence system. In this paper a brief review of dependence system theory precedes a more detailed discussion of some particular properties, e.g. the operator–image exchange property. Once again the duality of operators and resulting duality of properties of dependence systems (defined only when nontransitive operators are admitted), makes it possible to relate properties thus far studied in the context of separate mathematical theories.

## 1. Introduction

This paper is a successive stage in the study of dependence systems. The theory of dependence system, although still highly dependent on the context of investigations, has quite a long history, which can be traced at least from the early works of Schmidt [20] in the fifties, if not from the famous treatise of Moore [16] preceding even Hausdorff's 'Grundzüge der Mengenlehre', in which topology achieved its mature form. Indeed, a topological space was one of the earliest examples of dependence system. Anyway in the sixties the name 'abstract dependence theory' appeared in contexts of more or less limited generality [4, 21].

For a more comprehensive study of general dependence systems and the bibliography of the subject, I would like to refer to my other works [23–26], as this one in principle, concerns some specific aspects of dependence and is focused on problems which may seem to be unnecessarily complicated without reference to the development of the theory. However, in order to make the paper self-contained the first sections will provide the outline of ideas which form the background of the theory.

One of the main motivations for the attempts to create a unified theory of abstract dependence was the transfer of the methods used in the particular cases of dependence


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systems. Some of those systems such as the above-mentioned topological spaces and their generalizations, independence/dependence systems in algebra, matroids, have become fundamental for many branches of mathematics and have achieved their own theories with an immense stock of specific methods. Other theories, such as convexity (antimatroid) theory [7, 11] or logical consequence theory of Tarski [15, 27] are growing rapidly. Some theories can be observed *in statu nascendi*, as in the case of systems appearing in recursion theory [1], while only recently others are recognized as being members of the same family as in the case of von Neumann's continuous geometries, functional dependence [18], stability, absorption and kernels in graph theory [3], or the theory of bases in Banach spaces.

The most obvious example of the transfer of methods can be observed in the invasion of topology into many branches of mathematics. Linear dependence and free algebras indicate another direction of the interaction with the sources in classical and universal algebra [8]. Also matroid theory has interacted with the other branches of mathematics resulting for instance in combinatorial geometries.

The present paper contains results of the incorporation of combinatorial tools: Whitney's duality [29], which appeared so fruitful in graph theory, and Tutte's notion of the minor of a matroid [28], into general dependence theory. Unexpectedly these results appeared useful in some applications of the general theory to the particular case of dependence system in algebra. This transfer of combinatorial methods into algebra seems to be the most important aspect of the considerations presented in this paper.

One of the most effective characterizations (but not the only one possible) of a dependence system can be given by an operator — a slight generalization of the closure operator that is well known in particular cases of dependence systems such as topological spaces, matroids or of the generating operator of subalgebras of an algebra. In our general approach an operator on set $S$ is a function $f: 2^S \to 2^S$ satisfying only two of the three usually imposed conditions: (i) $\forall A \subseteq S$: $A \subseteq f(A)$, (ii) $\forall A, B \subseteq S$: $A \subseteq B \Rightarrow f(A) \subseteq f(B)$. The third condition (I): $\forall A \subseteq S: f(f(A)) = f(A)$, characterizes an important class of the so-called transitive (or some-times idempotent) dependence systems, which achieved great significance in mathematics and which found a wide range of applications. However in some cases the condition appears much too restrictive. Imposing some additional conditions on an operator, we can distinguish topological dependence systems, algebraic ones connected with the structure of subalgebras, matroids etc.

The classical problems arising when a particular dependence system is defined by some set of assumed properties are: existence of bases, equicardinality of bases (existence of dimension), cryptoisomorphic characterizations of the system, properties of objects which in some cases give cryptoisomorphic characterizations (such as the lattice of closed subsets in the case of the transitive operators, which in the general case provides only a partial description of the system).

Jones [14] considers algebraic closure operations satisfying an additional condition, called by him the *weak exchange property* (I will call it Jones's property (J) to avoid confusion with an entirely different property usually named this way).

The property can be identified as a special case of the more general characteristic of dependence systems: *operator-image exchange* (oiE) introduced in this paper and arising from attempts to weaken various known properties of dependence systems. This general setting makes it possible to examine relations of this condition to the other assumptions usually imposed on dependence systems of various specific forms depending on the numerous, highly differentiated applications. It concerns especially the conditions for the equicardinality of bases, with respect to which the oiE-operators are of special interest. It is shown that oiE, though it is an essential weakening of the standard conditions, when accompanied by an extremely weak form of idempotence (svswI) and the finite character property (fC), assures the equicardinality of bases. For the finite case the equicardinality of bases is given by oiE alone. The meaning of these facts appears clear recalling how surprising the result of Narkiewicz [17] was which stated the equicardinality of bases (if they exist) in v** -algebras (in the absence of the standard exchange properties) but certainly in the case of the transitive operator (I) of finite character (fC).

A natural question regarding the place of oiE among the other properties of dependence systems is answered by the identification of oiE as strengthening of the dual to the exchange property for finite independent sets (eP). Moreover the essential results of Jones [14] can be obtained as a corollary of the more general theorems concerning oiE. In turn the works of Jones [12–14] link the abstract theory to the concrete examples of application in semigroup theory.

What remains to be explained is the extent of generality of our formulation. The most general form of the theory achieved in the works of Dlab [6], where dependence is simply a relation between a set and its power set, seems to be inappropriate as it highly exceeds the needs of the study. However we cannot confine the generality to more than what was established in the formulation of Schmidt [20], fruitfully developed by Higgs [9, 10], Klee [15], Oxley [19] and others, without limiting the scope of application and loss of useful theoretical tools such as those mentioned above (Whitney's duality and Tutte's minor).

## 2. Preliminaries

In the following text the notation is inherited from the previously quoted classical papers and when necessary, a new one is patterned on it.

For the sake of simplicity we use the following rules for our notation: iff stands for 'if and only if' & for 'and'; capital letters stand for sets, lowercase letters for elements, one-element sets are denoted in the same way as their elements if no confusion is likely; $A \subset B$ iff $A \subseteq B$ and $A \neq B$; $|A|$ denotes the cardinality of the set $A$, $|A| < \omega$ means that $A$ is of a finite cardinality, $A^c = S \setminus A$ where $S$ is the universe of the discourse set.

Let $f$ be an operator on a set $S$ as it was defined above, an extensive and monotone function on the power set of $S$ which associates a subset $f(A)$ of $S$ with each subset $A$ of $S$ $\{f: 2^S \to 2^S,\ \forall A, B \subseteq S$: $A \subseteq f(A)$ & $[A \subseteq B \Rightarrow f(A) \subseteq f(B)]\}$. Then we can define the derived set operator $\mathrm{d}f$ of an operator $f$ by $\mathrm{d}f: 2^S \to 2^S$, $\mathrm{d}f: A \to A^{\mathrm{d}f} = \{x \in S: x \in f(A \setminus x)\}$.

There is a bijective correspondence between operators and derived set operators on a set $S$. For any function $d: 2^S \to 2^S$ satisfying (i) $A \subseteq B \Rightarrow d(A) \subseteq d(B)$, (ii) $x \in d(A)$ iff $x \in d(A \setminus x)$, there exists the unique operator $f$ such that $d(A) = A^{\mathrm{d}f}$ where $f(A) = A \cup d(A)$.

In what follows we will refer to some distinctive classes of subsets of $S$:

(i) $A \in f\text{-Cl} \subseteq 2^S$ iff $A^{\mathrm{d}f} \subseteq A$ iff $f(A) = A$ and we call $A$ an $f$-closed set or simply a closed set.

(ii) $A \in f\text{-Ind} \subseteq 2^S$ iff $A \cap A^{\mathrm{d}f} = \emptyset$ iff $\forall x \in A$: $x \notin f(A \setminus x)$ and we say: $A$ is $f$-independent or independent if no confusion is likely.

(iii) $A \in f\text{-Gen} \subseteq 2^S$ iff $A \cup A^{\mathrm{d}f} = S$ iff $S$ iff $f(A) = S$ and we call $A$ an $f$- generating set or simply a generating set.

(iv) $A \in f\text{-Base} \subseteq 2^S$ iff $A \in f\text{-Ind} \cap f\text{-Gen}$ iff $A^{\mathrm{d}f} = A^c$ and we call $A$ an $f$-base or simply a base.

The notion of a dual operator, defined below, has been fundamental in the successful development of dependence theory [22].

A *dual operator* $f^*$ for $f$ is defined by its derived set operator as follows: $f^*(A) = A \cup A^{\mathrm{d}f^*}$ where $A^{\mathrm{d}f^*} = [(A^c)^{\mathrm{d}f}]^c$. Certainly $f^{**} = f$.

We say properties $X$ and $Y$ of operators are dual if for each operator $f$ we have: $f$ has the property $X$ iff $f^*$ has the property $Y$. Then we write: $Y = X^*$ or $X = Y^*$.

Finally we define a *minor* of an operator $f$ on a set $S$ (a substructure operator generalizing both notions of restriction and contraction) for every $R$, $T$: $R \subseteq T \subseteq S$ by the following rule: $\forall A \subseteq T \setminus R$: $f^R_T(A) = f(A \cup R) \cap (T \setminus R)$. We say $f^R_T$ is a *closed minor* iff $T \in f\text{-Cl}$.

## 3. Properties of operators

As mentioned previously the particular cases of dependence systems can be distinguished by some additional conditions imposed on the operator which we assume merely to be an extensive and monotone function mapping the power set of the given set $S$ into itself. These additional properties and relations among them constitute the actual subject of our discussion. The full list of the properties attributed to the operators in all particular cases is too long to be placed here. I will recall only those which can make the way of description of the dependence systems more familiar to the reader and certainly those which will be in use in the sequel. For the convenience of the reader, summary of terms I use in this paper appears at the end of this section. The names of the conditions listed below in some cases are of long tradition in the literature of the subject. In the other cases I tried to follow the works in which the given condition appeared, which was not always possible because of differences in the terminology of some authors or because sometimes no names were given. In fact in the recent works the tendency to avoid full names can be observed, so in the sequel only abbreviations will be used. As a general rule for naming properties of operators (adopted from Klee [15]) in these abbreviations each capital denotes a singular property, lowercase letters preceding it are used for further specification (usually it will be weakening) e.g. wEsvswIfC is a conjunction of the three properties: E, I, C each specified by some small letters.

We can start from a review of some familiar cases.

A topological space can be described by INfA-operator (i.e. operator which has the properties I, N, fA), where:

- N means the normalization condition: $f(\emptyset)=\emptyset$,
- fA means the finite additivity: $\forall A, B\subseteq S$: $f(A\cup B)=f(A)\cup f(B)$,
- I is the transitivity (or idempotence) condition mentioned above which completes the set of assumptions for the standard closure operator. This condition can be formulated in a few equivalent ways for instance by the formulas, $\forall A, B\subseteq S$: $A\subseteq f(B) \Rightarrow f(A)\subseteq f(B)$, or $\forall A, B\subseteq S$: $A\subseteq f(B) \Rightarrow f(A\cup B)\subseteq f(B)$.

An algebraic generating operator (subalgebra operator) is IfC-operator, where fC means *finite character*: $\forall x\in S\forall A\subseteq S$: $x\in f(A) \Rightarrow \exists A_0\subseteq A$, $|A_0|<\omega$: $x\in f(A_0)$.

A matroid can be defined as an IwEfC-operator where wE is weak exchange: $\forall x, y\forall A\subseteq S$: $x\in f(A)$ & $x\in f(A\cup y) \Rightarrow y\in f(A\cup x)$.

The name *weak exchange* indicates the existence of more restrictive property called *exchange*: (E): $\forall x\in S$ $\forall A, B\subseteq S$: $x\in f(A)$ & $x\notin f(A\setminus B) \Rightarrow \exists y\in B$: $y\in f(A\setminus y\cup x)$.

Sometimes a matroid is defined in much weaker form with *very weak exchange*: (vwE): $\forall x\in S$ $\forall A, B\subseteq S$: $|B|<\omega$ & $f(A\cup x)=S$ & $x\in f(A)$ & $x\notin f(A\setminus B) \Rightarrow \exists y\in B$: $y\in f(A\setminus y\cup X)$.

A different kind of exchange property, well known from the early works on linear dependence and matroids, which played an essential role in the development of the theory, is *exchange property for finite independent sets* (eP): $\forall A, B\subseteq S$: $A, B\in f$-Ind & $1+|A|=|B|<\omega \Rightarrow \exists x\in B\setminus A$: $A\cup x\in f$-Ind. Usually in the finite case just this property is used in the definition of a matroid. But it has to be stressed, these properties in general are far from being equivalent or even similar, although in many works they appear with the same name — *exchange property*.

The notion of a geometry requires slightly more restrictive conditions. It is defined by IwEfCNt$_1$S-operator, where t$_1$S denotes the well-known $T_1$ condition from topology, also known as the separation axiom of Frechet: $\forall x\in S$: $f(x)=x$.

The transitivity condition can be weakened in many ways. Recall that transitivity condition can have a few equivalent forms. In some weakenings the equivalence is lost (only one side implication is always valid). There are two main schemes of the weakening dependent on which formula we choose from the two mentioned above. In each case weakening is obtained by limiting the range of the quantifiers in the formulas. The first scheme has its form: $\forall A, B\subseteq S$: $\Phi(A,B) \Rightarrow [A\subseteq f(B) \Rightarrow f(A\cup B)\subseteq f(B)]$, where:

- For I we can take simply: $\Phi(A,B)$ iff $A, B\subseteq S$.
- *Weak indempotence* wI is given by: $\Phi(A,B)$ iff $|A|<\omega$.
- *Very weak idempotence* (vwI) by: $\Phi(A,B)$ iff $|A\setminus B|<\omega$ & $B\in f$-Ind.

• *Singular very special weakened idempotence* (svwI) by: $\Phi(A,B)$ iff $|A|=1$ & $|B|<\omega$ & $A,B\in f$-ind.

The second scheme is: $\forall A,B\subseteq S$: $\Phi(A,B) \Rightarrow [A\subseteq f(B) \Rightarrow f(A)\subseteq f(B)]$, where:

*Weak transitivity* (wT) is given by the same $\Phi(A,B)$ as for wI i.e. $\Phi(A,B)$ iff $|A|<\omega$.

• *Restricted transitivity* (rT) by: $\Phi(A,B)$ iff $A,B\in f$-Ind.

If $xy$ represents some specification of the property I, then xyT follows from xyI as one of the implications of the equivalence I iff T remains in every weakening.

It appeared [10, 15] that the *weak exchange* is the dual property to the *weak idempotence* i.e. wE = wI* (also E = I*, vwE = vwI*). We will consider the condition W which is of great importance for the existence of bases. It has the following form [20]:

$$\forall A\subseteq S\ \forall x\in S\colon A\in f\text{-Ind} \quad\Rightarrow\quad [x\in A^{\mathrm{d}f} \text{ iff } A\cup x\notin f\text{-Ind}].$$

Finally we can formulate the *equicardinality of bases* as a property of an operator (ecB): $\forall A,B\subseteq S$: $A,B\in f$-Base $\Rightarrow |A|=|B|$. The expression $XY$-operator stands for an operator which has both the properties $X$ and $Y$ and as we will consider operators on a given set $S$ — our universe of discourse, we can assume the letters $X$, $Y$ represent the classes of operators which posses properties $X$, $Y$ respectively. So in the sequel I will use the shortest form of assertion that an operator $f$ has the property $X$: $f\in X$. Obviously $f\in XY$ means of $f\in X\cap Y$. It also makes possible to express the fact that the property $X$ is stronger than $Y$ (or property $X$ implies $Y$) by $X\subseteq Y$ i.e. as the inclusion of the class $X$ of operators in the class $Y$. For example we can write the fundamental relations among weakened forms of the transitivity: I ⊆ wI ⊆ vwI ⊆ svswI, I ⊆ wT, I ⊆ rT but neither rT ⊆ wI nor wI ⊆ rT [2, 15]. Certainly we have: $X\subseteq Y \Rightarrow X^*\subseteq Y^*$. There are some other standard inclusions which play an essential role in the theory [15]:

(i) vwIfC ⊆ I
(ii) wEvwIfC ⊆ IE
(iii) wEIfC ⊆ ecB.

In the proof of Corollary 6.3 we will use the standard inclusion from finite matroid theory (i.e. matroid on a set $S$, such that $|S|<\omega$): eP ⊆ ecB [28].

All classes of operators on a given set $S$ on which we will operate in the sequel are listed below to make the access to their definitions easy:

(ecB): $\forall A,B\subseteq S$: $A,B\in f$-Base $\Rightarrow |A|=|B|$.
(fC): $\forall x\in S\ \forall A\subseteq S$: $x\in f(A) \Rightarrow \exists A_0\subseteq A, |A_0|<\omega$: $x\in f(A_0)$.
(E): $\forall x\in S\ \forall A,B\subseteq S$: $x\in f(A)$ & $x\notin f(A\setminus B) \Rightarrow \exists y\in B$: $y\in f(A\setminus y\cup x)$.
(wE): $\forall x,y\ \forall A\subseteq S$: $x\notin f(A)$ & $x\in f(A\cup y) \Rightarrow y\in f(A\cup x)$.
(vwE): $\forall x\in S\ \forall A,B\subseteq S$: $|B|<\omega$ & $f(A\cup x)=S$ & $x\in f(A)$ & $x\notin f(A\setminus B) \Rightarrow \exists y\in B$: $y\in f(A\setminus y\cup x)$.
(I): $\forall A,B\subseteq S$: $A\subseteq f(B) \Rightarrow f(A\cup B)\subseteq f(B)$.
(T=I): $\forall A,B\subseteq S$: $A\subseteq f(B) \Rightarrow f(A)\subseteq f(B)$.
(wI): $\forall A,B\subseteq S$: $|A|<\omega \Rightarrow [A\subseteq f(B) \Rightarrow f(A\cup B)\subseteq f(B)]$.
(vwI): $\forall A,B\subseteq S$: $|A\setminus B|<\omega$ & $B\in f$-Ind $\Rightarrow [A\subseteq f(B) \Rightarrow f(A\cup B)\subseteq f(B)]$.

(svswI): $\forall A, B \subseteq S: |A| = 1 \;\&\; |B| < \omega \;\&\; A, B \in f\text{-Ind} \Rightarrow [A \subseteq f(B) \Rightarrow f(A \cup B) \subseteq f(B)]$.
(rT): $\forall A, B \subseteq S$: $A, B \in f\text{-Ind} \Rightarrow [A \subseteq f(B) \Rightarrow f(A) \subseteq f(B)]$.
(eP): $\forall A, B \subseteq S$: $A, B \in f\text{-Ind} \;\&\; 1 + |A| = |B| < \omega \Rightarrow \exists x \in B \setminus A$: $A \cup x \in f\text{-Ind}$.
(W): $\forall A \subseteq S \; \forall x \in S$: $A \in f\text{-Ind} \Rightarrow [x \in A^{df}$ iff $A \cup x \notin f\text{-Ind}]$.

## 4. Operator–image exchange property

At this point the necessary foundations are in place so that we can turn to the subject o this paper, operator–image exchange operators.

We say that an operator $f$ on a set $S$ is an oiE-operator if it satisfies the following condition:

$$\text{(oiE)}: \forall B, C \subseteq S,\; B \neq \emptyset: [f(B) = f(C) \Rightarrow \forall x \in C \exists y \in B: f(B) = f(C \setminus x \cup y)].$$

**Lemma 4.1.** *If $f$ is an* oiE*-operator on $S$, then:*

$$\forall B, C \subseteq S: B, C \in f\text{-Ind} \;\&\; |B| < \omega \;\&\; |C| < \omega \;\&\; f(B) = f(C) \Rightarrow |B| = |C|.$$

**Proof.** We will show that if $n = |B| \geqslant |C| = m$, then $n = m$. Let $n > m$, $C = \{x_1, \ldots, x_m\}$, $B = \{y_1, \ldots, y_n\}$, then by oiE, $f(B) = f(C \setminus x_1 \cup y_{i1})$ for some $y_{i1} \in B$. Now we adopt the following notation: $C_0 = C$, $C_1 = C \setminus x_1 \cup y_{i1}$ so $f(B) = f(C_1)$ and by oiE: $f(B) = f(C_1 \setminus x_2 \cup y_{i2})$ for some $y_{i2}$ not necessarily different from $y_{i1}$.

Now $C_k = C_{k-1} \setminus x_k \cup y_{ik}$; so $f(B) = f(C_{k-1})$ implies $f(B) = f(C_k)$. Certainly $C_m = \{y_{i1}, \ldots, y_{im}\}$ and finally $f(B) = f(C_m)$, but $m < n$ so $C_m \subset B$, therefore $B \notin f\text{-Ind}$, a contradiction. □

**Remark 4.2.** The proof is valid for an extended version of the lemma that is

$$\text{oiE} \Rightarrow \forall B, C \subseteq S: |C| < \omega \;\&\; B \in f\text{-Ind} \;\&\; f(B) = f(C) \Rightarrow |B| \leqslant |C|.$$

**Lemma 4.3.** *If $f$ is an* oiEsvswIfC*-operator, then: $\forall A \subseteq S$, $A$ an infinite set $\forall B \subseteq S$: $A, B \in f\text{-Ind} \;\&\; f(A) = f(B) \Rightarrow |A| = |B|$.*

**Proof.** It suffices to show that $|A| \leqslant |B|$. Let $|B| < |A|$; so by fC we have $\forall x \in B \; \exists A_x \subseteq A$: $|A_x| < \omega \;\&\; x \in f(A_x)$. *Now let* $A_B = \cup \{A_x : x \in B\} \subseteq A$, so both $A$ and $B$ are finite (hence their cardinalities are less than $|A|$) or $|A_B| = |B| < |A|$, and in both cases $B \subseteq f(A_B)$.

Let $a \in A \setminus A_B \neq \emptyset$, then by oiE $\exists b \in B$: $f(A \setminus a \cup b) = f(A)$ and $b \in f(A_B) \subseteq f(A \setminus a)$.

Now $a \in f(A) = f(A \setminus a \cup b)$, so $\exists A_a \subseteq A \setminus a$: $|A_a| < \omega \;\&\; a \in f(A_a \cup b)$ and $\exists A_b \subseteq A \setminus a$: $|A_b| < \omega \;\&\; b \in f(A_b)$, therefore $b \in f(A_a \cup A_b) \;\&\; |A_a \cup A_b| < \omega \;\&\; A_a \cup A_b \in f\text{-Ind}$ and by svswI: $f(A_a \cup A_b \cup b) = f(A_a \cup A_b)$, so $a \in f(A_a \cup A_b) \subseteq f(A \setminus a)$, which means that $A \notin f$-Ind, a contradiction. Therefore $|A| \leqslant |B|$. □

**Remark 4.4.** The following example shows that there exist svswIfC-operators which are not IfC-operators (i.e. svswIfC $\not\subseteq$ IfC). Therefore Lemma 4.3 is actually a strengthening of the statement that for every IfC-operator infinite bases are of the same cardinality.

Let $S = \mathbb{N}$ (set of positive integers), $f: 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, such that $f(\emptyset) = \{1\}$, $f(1) = \{m \in \mathbb{N}: \exists k \in \mathbb{N}: m = 2k - 1\}$, $f(A) = \{1, 2, \ldots, \max A\}$ if $1 \notin A$ & $|A| < \omega$, $f(A) = \mathbb{N} = S$ if $1 \in A$ or $A$ is infinite.

Then it is easy to prove $f$ is a svswIfC-operator, but for every finite $A$, such that $1 \notin A$ we have: $f(A \cup 1) = S \not\subseteq f(A) = \{1, 2, \ldots, \max A\}$ and certainly $1 \in f(A)$, hence $f$ is not an I-operator.

As a corollary of the preceding lemmas we get the following theorem.

**Theorem 4.5.** *If $f$ is* oiEsvswIfC-*operator on $S$, then each closed restriction of $f$ has* ecB *property.*

**Remark 4.6.** It follows from the theorem that oiEsvswIfC $\subseteq$ ecB.

## 5. Relations to the other properties

As long as we do not have W property, $f$ can have no bases at all, so Theorem 4.5 states that in each closed restriction bases have the same cardinality provided they exist. It is interesting to know what are the relations of oiE property to the other properties characterizing dependence systems, especially to W property which accompanied by fC assures existence of bases for each restriction and also to the other properties which imply ecB. The following propositions answer this question.

**Proposition 5.1.** (i) oiEIfC $\not\subseteq$ eP, (ii) oiEfC $\not\subseteq$ vwE, (iii) oiEIfC $\not\subseteq$ W.

**Proof.** By an example of oiEIfC-operator $f$ which is neither eP, nor vwE, nor W.

Let $f(A) = A$ if $A \cap T = \emptyset$ and $f(A) = S$ otherwise, for some $T, \emptyset \subset T \subset S$ and every $A \subseteq S$. Certainly $f$ is an I-operator, but it is not wE-operator which can be seen when we take $x \notin A \cup T$ & $y \in T$, so $x \notin f(A)$ & $x \in f(A \cup y)$ but $y \notin f(A \cup x)$. Now let $x \in T$ & $y$, $z \notin T$, $y \neq z$, then $\{x\} \in f$-Ind & $\{y, z\} \in f$-Ind but $\{x, z\} \notin f$-Ind, as well as $\{x, y\} \notin f$-Ind, so $f$ is not an eP-operator. It is fC-operator, so it can not be vwE-operator, otherwise as IvwEfC-operator it would be wE. By a similar argument it can not be W-operator.

Now we should show that it has oiE property. If $f(A) = f(B) \neq S$, then $A = B$. If $f(A) = f(B) = S$, then $\forall x \in A$: $x \notin T \Rightarrow f(A \setminus x) = f(A) \Rightarrow \forall y \in B$: $f(A \setminus x \cup y) = f(B)$. If $x \in T$ and $f(A \setminus x) = A \setminus x$, then $\forall y \in B \cap T$: $f(A \setminus x \cup y) = f(B)$. $\square$

**Proportion 5.2.** oiE $\not\subseteq$ vwI.

**Proof.** (by an example) Let $T\subseteq S$, $T\neq\emptyset$, $|T|<\omega$ and $\forall A\subseteq S$: $A\cap T=\emptyset \Rightarrow f(A)=A$ & $\emptyset\neq A\cap T\neq T \Rightarrow f(A)=A\cup T$ & $T\subseteq A \Rightarrow f(A)=S$. We have $A\in f$-Ind iff $|A\cap T|\leqslant 1$, so $f$ is not vwI-operator, as can be seen for $B\cup T\neq S$ & $B\cap T=\{x\}$, which implies: $f(B)=B\cup T$ & $B\in f$-Ind & $|T\setminus B|<\omega$, so $T\setminus B\subseteq f(B)=B\cup T$, but $f[B\cup(T\setminus B)]=f(T)=S\neq B\cup T$. Certainly it is oiE-operator. □

**Proposition 5.3.** eP$\not\subseteq$oiE.

**Proof.** (by an example) Let $f(A)=A$ if $T\not\subseteq A$ & $R\not\subseteq A$ for some $R,T\not\subseteq S, R\cap T=\emptyset$, $R$ and $T$ nonempty and infinite, $f(A)=S$ otherwise. Certainly $f$ is an eP-operator because $\forall A\subseteq S$: $|A|<\omega \Rightarrow A\in f$-Ind, but it is not oiE-operator as can be seen in the following: $f(R)=f(T)$, but if $x\in R$, then $\forall y\in T$: $f(R\setminus x\cup y)= R\setminus x\cup y\neq f(T)=S$. □

**Proposition 5.4.** IwE$\subseteq$oiE.

**Proof.** Let $C_x=C\setminus x$ and $f(B)=f(C)$, then we have to show that $\exists y\in B$: $f(B)=f(C_x\cup y)$ but having by I: $f(C_x\cup y)\not\subseteq f(B)$, actually that $\exists y\in B$: $f(B)\subseteq f(C_x\cup y)$. If $f(C_x)=f(B)$ then by wI it can be seen that $\forall y\in B$: $f(B)=f(C_x\cup y)$.

If $f(C_x)\subset f(B)$, then $B\subseteq f(C_x\cup x)$ and by I: $B\not\subseteq f(C_x)$, since $\exists y\in B$: $y\in f(C_x\cup x)\setminus f(C_x)$ and by wE: $x\in f(C_x\cup y)$ therefore $C_x\cup x\subseteq f(C_x\cup y)$ and by I: $f(B)=f(C_x\cup x)\subseteq f(C_x\cup y)$. □

**Remarks 5.5.** Although oiE is weaker than the standard properties characterizing ecB-operators (IwE), there is no simple relation of it to the other properties usually taken into account in the absence of fC condition.

## 6. Dual operator–image exchange property

Seeking the place of oiE property in characterizations of dependence systems it is worth-while to examine its dual property.

**Lemma 6.1.** *The dual of* oiE *has the following form*:

$$\text{oiE}^*\text{: } \forall A\, B,\, C\subseteq S\text{: } B\neq S \;\&\; B\cap B^{\mathrm{d}f}=C\cap C^{\mathrm{d}f}$$

$$\Rightarrow(\forall x\notin C\,\exists y\notin B\text{: } [(C\cup x)\setminus y]\cap[(C\cup x)\setminus y]^{\mathrm{d}f}=B\cap B^{\mathrm{d}f}).$$

**Proof.** $f(B)=f(C)$ iff $B\cup B^{\mathrm{d}f}=C\cup C^{\mathrm{d}f}$ iff $B^{\mathrm{c}}\cap(B^{\mathrm{c}})^{\mathrm{cd}f\mathrm{c}}=C^{\mathrm{c}}\cap(C^{\mathrm{c}})^{\mathrm{cd}f\mathrm{c}}$ iff $B^{\mathrm{c}}\cap B^{\mathrm{cd}f*}=C^{\mathrm{c}}\cap C^{\mathrm{cd}f*}$. Therefore: oiE iff $(\forall B, C\subseteq S$: $[B\neq\emptyset$ & $B^{\mathrm{c}}\cap B^{\mathrm{cd}f*}=C^{\mathrm{c}}\cap C^{\mathrm{cd}f*}]$

$\Rightarrow \forall x \in C \exists y \in B$: $[(C\setminus x)\cup y]^c \cap [(C\setminus x)\cup y]^{cdf*} = B^c \cap B^{cdf*})$ iff $\{\forall B, C \subseteq S$: $B \neq S$ & $B \cap B^{df*} = C \cap C^{df*} \Rightarrow \forall x \notin C \exists y \notin B$: $[(C\cup x)\setminus y] \cap [(C\cup x)\setminus y]^{df*} = B \cap B^{df*}\}$. So by the substitution $f$ for $f^*$ we get oiE*. □

**Lemma 6.2.** oiE* $\subseteq$ eP.

**Proof.** First we notice that oiE* is equivalent to the following: $\forall B, C \subseteq S$: $B \neq S$ & $B \cap B^{df} = C \cap C^{df} \Rightarrow \forall x \in B \setminus C \exists y \notin B$: $[(C\cup x)\setminus y] \cap [(C\cup x)\setminus y]^{df} = B \cap B^{df}$, which is evident because if $x \notin B$ then $y$ can be simply $x$.

Now we observe that as $A \in f$-Ind iff $A \cap A^{df} = \emptyset$, oiE* implies: $\forall B, C \in f$-Ind $\forall x \in B \setminus C$ $\exists y \notin B$: $(C\cup x)\setminus y \in f$-Ind, and recall that

$$\text{eP: } \forall B, C \subseteq S\text{: } B, C \in f\text{-Ind } \&\ 1 + |B| \leqslant |C| < \omega \ \Rightarrow\ \exists x \in C \setminus B\text{: } B \cup x \in f\text{-Ind}.$$

Now let $B \cap C = \{z_1, \ldots, z_m\}$ & $B \setminus C = \{x_1, \ldots, x_n\}$ & $C \setminus B = \{y_1, \ldots, y_{n+1}\}$. By oiE* and the implication mentioned above $\exists y \notin B$: $(C \cup x_1)\setminus y \in f$-Ind.

Let $C_1$ stand for $(C\cup x_1)\setminus y$, then $1 + |B| \leqslant |C_1|$ (equality if $y \in C$) and we can repeat operation with $C_1$ instead of $C$ getting $C_2$ ($x_1 \in B \cap C_1$).

After $n$ successive steps we get $C_n \supseteq B$, $|B| + 1 = m + n + 1 \leqslant |C_n| \leqslant m + 2n + 1$, $C_n \in f$-Ind and $\exists x \in C_n \setminus B$: $B \cup x \in f$-Ind, but $C_n \subseteq B \cup C$, so we get eP. □

**Corollary 6.3.** *If $S$ is a finite set and $f$ is* oiE-*operator, then $f^*$ has its bases of equal cardinality and therefore $f$ has its bases of equal cardinality.*

**Proof.** In general an eP-operator $f$ on a finite set $S$ has ecB property. Hence $f^*$ has ecB property. The result follows from the fact that bases of $f^*$ are set complements in $S$ of bases of $f$. But no such a reasoning remains true if $S$ is infinite. □

Summarizing we can state the following theorem.

**Theorem 6.4.** oiE *is strengthening of the dual to* eP.

## 7. Closed minors of oiE-operators

**Lemma 7.1.** *If $f$ is* oiEwI-*operator on $S$, then each closed minor of $f$ is an* oiEwI-*operator.*

**Proof.** Certainly each minor of wI-operator is wI-operator [9]. Let $R \subseteq T \subseteq S$ and $f_T^R$ be a closed minor of $f$. $f_T^R(B) = f_T^R(C)$ iff $f(B\cup R) = f(C\cup R)$ $\Rightarrow$ $\forall x \in C$ $\exists y \in B \cup R$: $f(B\cup R) = f[(C\cup R)\setminus x \cup y]$, but if $y \in R$ then $f[(C\cup R)\setminus x \cup y] = f(C\setminus x \cup R) = f(B\cup R)$ and $\forall z \in B$: $z \in f(C\setminus x \cup R)$, so by wI: $\forall z \in B$: $f(C\setminus x \cup z \cup R) = f(B\cup R)$ and we get:

$\forall x \in C \exists y \in B$: $f(C \setminus x \cup y \cup R) = f(B \cup R)$ and therefore $\forall x \in C\, \exists y \in B$: $f_T^R(C \setminus x \cup y) = f_T^R(B)$. □

**Theorem 7.2.** *If f is an* oiEvwIfC-*operator on S then each closed minor of f is an* ecB-*operator.*

**Proof.** Recalling that vwIfC ⊆ wIfC ⊆ IfC [10], the theorem follows immediately from Theorem 4.5 and Lemma 7.1. □

## 8. Jone's property

Jones [14] examined IfC-operators with an additional property called by him 'weak exchange':

$$\mathrm{J}:\ \forall A, B \subseteq S\ \forall x \in S: f(B) = f(A \cup x) \Rightarrow \exists y \in B: x \in f(A \cup y).$$

The main result of Jones is that in the class of IfC-operators J is equivalent to equicardinality of $D$-bases in each $D$-closed sub-set $A$ of $S$ [which means with respect to an operator ${}_Df$, where ${}_Df(A) = f(A \cup D)$], which in fact can be interpreted as equicardinality of bases in each closed minor. The more essential implication of this theorem follows immediately from Theorem 7.2, as Jone's operators are identical with oiEIfC-operators, as can be seen in the following proposition.

**Proposition 8.1.** oiE ⊆ J and JI = oiEI.

**Proof.** Certainly J is equivalent to the following condition:

$$\forall B, C \subseteq S: f(B) = f(C) \Rightarrow \forall x \in C\ \exists y \in B: x \in f(C \setminus x \cup y),$$

so oiE ⊆ J is evident, and also oiEI ⊆ IJ.

Now $x \in f(C \setminus x \cup y)$ & $C \setminus x \subseteq f(C \setminus x \cup y) \Rightarrow C \subseteq f(C \setminus x \cup y)$ and by I we have $f(C) \subseteq f(C \setminus x \cup y)$. If $y \in B$, then $y \in f(C)$ and $C \setminus x \cup y \subseteq f(C)$, so by I: $f(C \setminus x \cup y) \subseteq f(C)$ and therefore $f(B) = f(C) = f(C \setminus x \cup y)$. □

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